TERES - Tail Event Risk Expectile based Shortfall

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Motivation

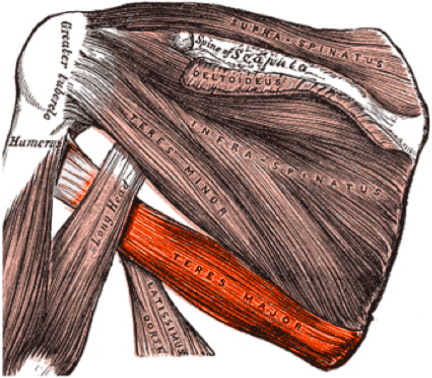

Figure 1: The teres major muscle

Tail Risk

Figure 2: Nezha (link)

VaR and ES

- Value at Risk (VaR)
  - Basel III
  - Not coherent ▸ Coherence
- Expected Shortfall (ES) ▸ ES Definition
  - Basel Committee (2014)
  - Coherent, focus on tail structure

Example: Deutsche Bank - risk levels $\{0.0002, 0.001, 0.01\}$, Kalkbrenner et al. (2014)

Quantile VaR

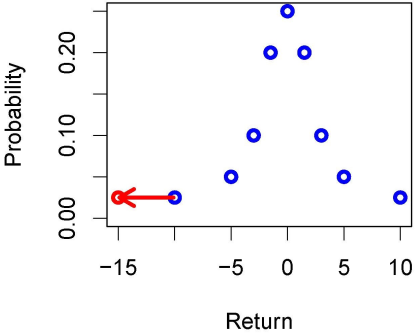

Figure 3: Distribution of returns, $\widehat{VaR}_{0.05}$ remains unchanged under changing tail structure, clouding the investors risk perception

Objectives

(i) Understanding Expected Shortfall (ES)

- Extreme events and associated risk
- Distributional environments - implications

(ii) TERES

- Tail driven risk assessment, robustness of ES
- Advantages of expectiles

Tail Risk

Example: 2008 subprime mortgage crisis

- S&P 500 long position in 2008 (261 daily returns)
- Quantification of the 1% portfolio risk

Scenario analysis, $\mu = -0.002, \sigma = 0.025$

- Scenario (i) Normal distribution
  VaR $= -5.9\%$, *ES* $= -6.8\%$
- Scenario (ii) Laplace distribution
  VaR $= -10.0\%$, *ES* $= -12.5\%$

Tail Risk

	2008	2010	2012	2014
Normal distribution				
VaR	-5.9	-2.6	-1.8	-1.6
ES	-6.8	-3.0	-2.1	-1.9
Laplace distribution				
VaR	-10.0	-4.4	-3.1	-2.4
ES	-12.5	-5.6	-3.9	-3.5

Estimated VaR and ES in % for S&P 500 index returns at level $\alpha = 0.01$

Research Questions

What are the thrills for ES estimation?

What are the robustness properties of ES?

Which risk range is expected under different tail scenarios?

Outline

1. Motivation ✓
2. Expected Shortfall
3. TERES
4. Empirical Results
5. Conclusions

Expected Shortfall

- Financial returns Y
  - pdf $f(y)$ and cdf $F(y)$
  - Here: lower tail (downside) risk

- Expected shortfall

$$s_\eta = \mathsf{E}[Y|Y < \eta]$$

  - Basel: Value at Risk threshold $\eta = q_\alpha = F^{-1}(\alpha)$

Expectiles

- ES estimation
  - Using expectiles, Taylor (2008)
  - Expectiles reflect the tail structure

- Loss function

$$\rho_{\alpha,\gamma}(u) = |\alpha - \mathbf{I}\{u < 0\}|\,|u|^{\gamma} \tag{1}$$

  - Expectile $e_\alpha = \arg\min_{\theta} \mathsf{E}\,\rho_{\alpha,2}(Y - \theta)$
  - Quantile $q_\alpha = \arg\min_{\theta} \mathsf{E}\,\rho_{\alpha,1}(Y - \theta)$

Loss Function

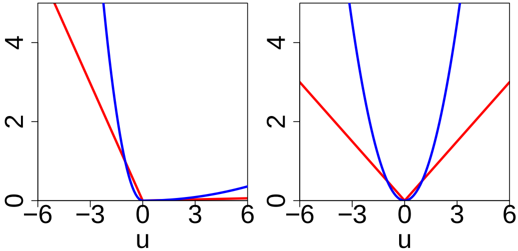

Figure 4: Expectile and quantile loss functions at $\alpha = 0.01$ (left) and $\alpha = 0.50$ (right)

LQRcheck

Tail Structure

- Quantiles and expectiles - one-to-one mapping ▸ Details
  - Goal: $e_{w(\alpha)} = q_\alpha$
  - Find expectile level $w(\alpha)$

- ES using expectiles, Taylor (2008) ▸ Proof

$$s_{q_\alpha} = e_{w(\alpha)} + \frac{e_{w(\alpha)} - \mathsf{E}[Y]}{1 - 2w(\alpha)} \frac{w(\alpha)}{\alpha}$$

Expectiles and Quantiles

- Jones (1993), Guo and Härdle (2011)
  - Analytical formula for level $w(\alpha)$ ▸ Details
  - Assumption: known return distribution $F(\cdot)$

Example, $\mathsf{N}(0,1)$

$$w(\alpha) = \frac{-\varphi(q_\alpha) - q_\alpha \alpha}{-2\{\varphi(q_\alpha) + q_\alpha \alpha\} + q_\alpha - \mathsf{E}[Y]}$$

Sensitivity Analysis

- For the more general framework given in (1) ▸ Proof

$$w(\alpha,\gamma) = \frac{\int_{-\infty}^{q_\alpha} |y-q_\alpha|^{\gamma-1} dF(y)}{\int_{-\infty}^{\infty} |y-q_\alpha|^{\gamma-1} dF(y)}, \quad \gamma \geq 1$$

- Quantile case $w(\alpha,1)=\alpha$, convergence in γ
  - $|y-q_\alpha|>1$: exponential convergence towards α as $\gamma \to 1$
  - $|y-q_\alpha|<1$: root convergence

Heavy Tailed Returns

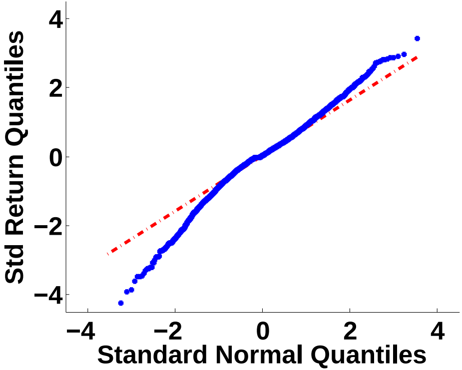

Figure 5: S&P 500 return quantiles from 20050103-20141231, standardized using a GARCH(1,1) model

TERES_Standardization

Quantile - Expectile Relation

- Properties of ES
  - ES depends on the e_α to q_α distance
  - Other component: VaR
- Implications of thickening the tail
  - Expectile-quantile relation given a distribution
  - **Examples:** Normal and Laplace case

Quantile - Expectile - Normal

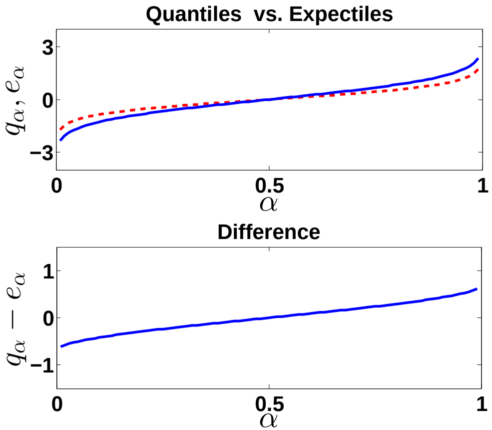

Figure 6: Top: Quantile (blue) and Expectile (red), bottom: difference

Quantile - Expectile - Laplace

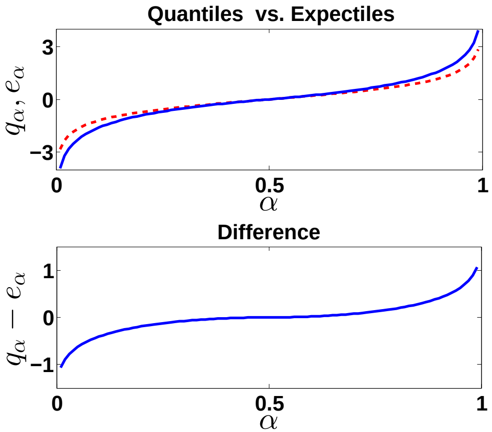

Figure 7: Top: Quantile (blue) and Expectile (red), bottom: difference

Quantile - Expectile - Equality

- Is there a distribution such that $e_\alpha = q_\alpha$
- Consider a (very heavy tailed) cdf, Koenker (1993)

$$F(x) = \begin{cases} 0.5 - 0.5\left(1 - \frac{4}{4+x^2}\right)^{0.5}, & \text{if } x < 0 \\ 0.5 + 0.5\left(1 - \frac{4}{4+x^2}\right)^{0.5}, & \text{else} \end{cases}$$

Quantile - Expectile - Equality

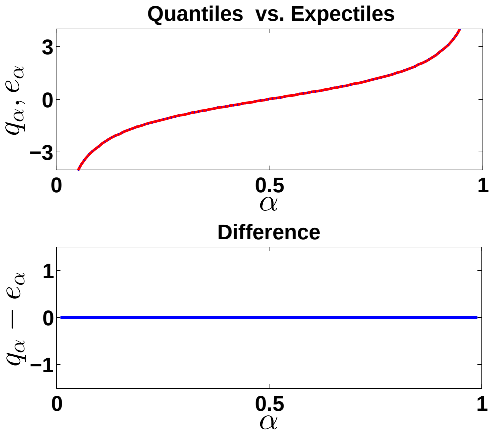

Figure 8: Top: Quantile (blue) and Expectile (red), bottom: difference

TERES

- Flexible statistical framework - ES tail scenarios
  - Properties of ES in an environment
  - Risk corridor, scenario analysis
- Family of distributions - environment
  - Distributional families, e.g. exponential
  - Mixtures, e.g. two-component linear mixture

Example: Contaminated Normal Environment

- $\delta-$environment, Huber (1964)

$$f_\delta(y) = (1-\delta)\varphi(y) + \delta h(y), \quad \delta \in [0,1]$$

- Practice: normality assumption, findings: heavy tails
  - Financial markets: $h(\cdot)$ is symmetrical and heavy tailed
  - Contamination degree δ

Example: Normal-Laplace Mixture

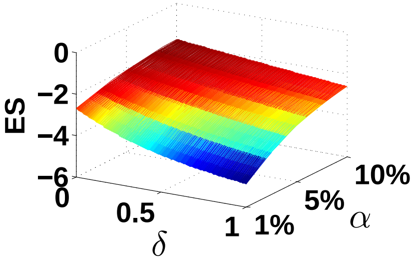

Figure 9: Theoretical ES_{q_α} for different contamination δ and risk level α

TERES_ES_Analytical

Relation to MLE

- Asymmetric Generalized Error Distribution (AGED), Ayebo and Kozubowski (2003)

$$
\begin{aligned}
f(x) =& \frac{\gamma}{\sigma\Gamma(\frac{1}{\gamma})} \frac{\kappa}{1+\kappa^2} \exp\left\{ \left(-\frac{\kappa^\gamma}{\sigma^\gamma} \mathbf{I}\{x-\mu \geq 0\} \right.\right. \\
& \left.\left. - \frac{1}{\kappa^\gamma \sigma^\gamma} \mathbf{I}\{x-\mu < 0\} \right) |x-\mu|^\gamma \right\} \\
\Gamma(x) =& \int_0^\infty x^{t-1} \exp(-x) dx
\end{aligned}
$$

- Scale σ, skewness κ, location μ and shape γ (Asymmetric Laplace $\gamma = 1$ and normal $\gamma = 2$)

M-Quantiles as Location Estimate

- AGED Log-likelihood

$$
\begin{aligned}
-\log\{f(x|\mu,\sigma,\gamma,\kappa)\} =& c(\gamma,\sigma,\kappa) + \left(\frac{\kappa^\gamma}{\sigma^\gamma}\mathbf{I}\{x-\mu \geq 0\}\right. \\
&\left. + \frac{1}{\kappa^\gamma\sigma^\gamma}\mathbf{I}\{x-\mu < 0\}\right)|x-\mu|^\gamma
\end{aligned}
$$

- The location MLE μ^{OPT} is equal to the τ-M-Quantile if

$$
\kappa(\delta) = \left(\frac{\tau}{1-\tau}\right)^{\frac{1}{2\delta}}
$$

Financial Applications

Stock returns
DAX, FTSE 100 and S&P 500
Different risk levels

▸ Stock Example
▸ Intraday Margin

Foreign Exchange
EUR/UAH exchange rate
Not relying on standardization

▸ Forex Example

Portfolio Selection
TEDAS (Tail Event Driven ASset allocation)
Small sample size

▸ TEDAS Example

Data

- DAX, FTSE 100 and S&P 500 daily returns
  - Risk level α: 0.01, 0.05 and 0.10
  - Varying tail thickness δ
- Span: 20050103-20141231 (2609 trading days)
  - One-year time horizon (250 trading days) - moving window
  - Standardized returns

Financial Applications

Returns

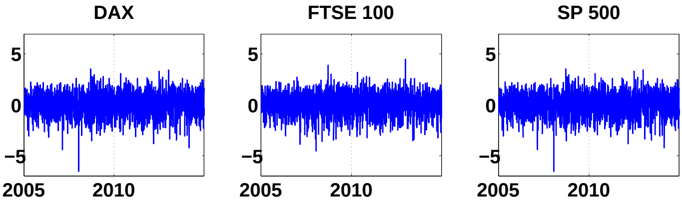

Figure 10: Standardized returns of the selected indices from 20050103-20141231

Financial Applications

TERES_Standardization

Expected Shortfall

δ	DAX	FTSE 100	S&P 500
0.0	-2.91	-3.11	-3.26
0.001	-2.91	-3.11	-3.26
0.002	-2.91	-3.12	-3.27
0.005	-2.92	-3.13	-3.28
0.01	-2.94	-3.14	-3.30
0.02	-2.97	-3.17	-3.33

Table 1: Estimated ES_{q_α} for selected indices at $\alpha = 0.01$, from 20140116-20141231 (250 trading days)

Financial Applications

TERES_RollingWindow

Expected Shortfall

δ	DAX	FTSE 100	S&P 500
0.05	-3.05	-3.26	-3.42
0.1	-3.16	-3.38	-3.54
0.15	-3.24	-3.46	-3.63
0.25	-3.32	-3.55	-3.72
0.5	-3.30	-3.53	-3.70
1.0	-3.19	-3.41	-3.57

Table 2: Estimated ES_{q_α} for selected indices at $\alpha = 0.01$, from 20140116-20141231 (250 trading days)

▸ Financial Applications

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ES Dynamics

Setup

- Risk level α: 0.10, 0.05 and 0.01
- Scenarios: Laplace and normal

Empirical Study

- Rolling window exercise - one-year time horizon (250 days)
- Stock markets: German, UK, US

▸ Financial Applications

ES Dynamics

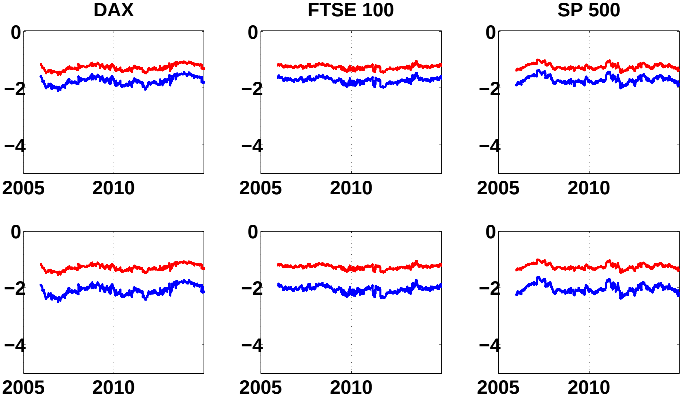

Figure 11: ES_{q_α} and VaR at $\alpha = 0.10$; $\delta = 0$ (top) and $\delta = 1$ (bottom)

Financial Applications

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ES Dynamics

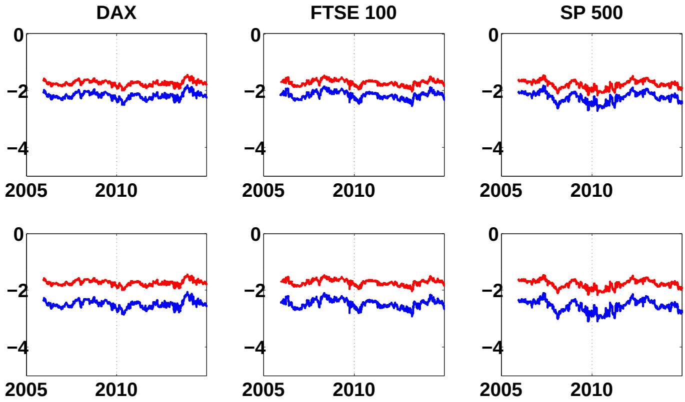

Figure 12: ES_{q_α} and VaR at $\alpha = 0.05$; $\delta = 0$ (top) and $\delta = 1$ (bottom)

Financial Applications

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ES Dynamics

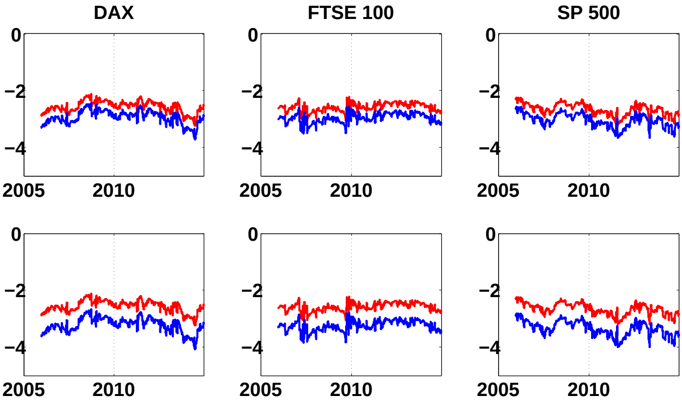

Figure 13: ES_{q_α} and VaR at $\alpha = 0.01$; $\delta = 0$ (top) and $\delta = 1$ (bottom)

▸ Financial Applications

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Intraday Margin

Example: Finance - portfolio exposure

An investor enters a 1 Mio USD long position (e.g., S&P 500) on 20141231.

Using the last 250 standardized returns, the rescaled ES is obtained as

ES_{q_α} in USD	$\alpha = 0.05$	$\alpha = 0.01$	$\alpha = 0.005$
$\delta = 0$	-10,358	-16,694	-19,188
$\delta = 1$	-11,880	-18,319	-20,821

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Intraday Margin

Example: Finance - portfolio exposure

An investor enters a 1 Mio USD long position (e.g., S&P 500) at the height of the financial crisis (20071101).

Using the last 250 standardized returns, the rescaled ES is obtained as

ES_{q_α} in USD	$\alpha = 0.05$	$\alpha = 0.01$	$\alpha = 0.005$
$\delta = 0$	-143,122	-185,941	-194,738
$\delta = 1$	-162,529	-203,008	-210,432

Financial Applications

TERES_RollingWindow

Exchange Rate Application

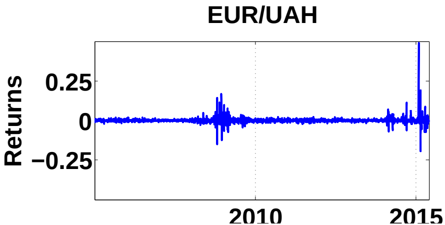

Figure 14: Returns of the EUR/UAH exchange rate from the 20050103 to 20150601

Financial Applications

ES Dynamics

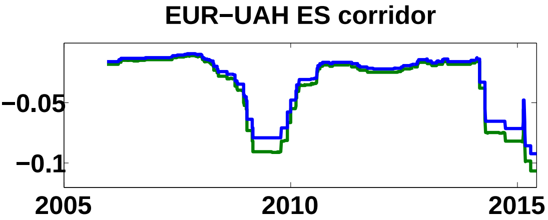

Figure 15: $\mathrm{ES}_{q_{0.01}}$ Normal-Laplace risk extrema (corridor). Normal $\delta = 0$ and "worst case", i.e. $\delta = \frac{1}{3}$ scenarios using a rolling window of 250 obs.

▸ Financial Applications

TERES_RollingWindow

ES Dynamics

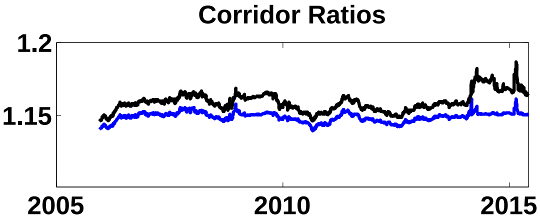

Figure 16: Ratio of the minimal and maximal risk indications in a Normal-Laplace environment using risk levels $\alpha = 0.01$ (blue) and $\alpha = 0.05$ (black)

Financial Applications

TERES_RollingWindow

Portfolio (TEDAS) Application

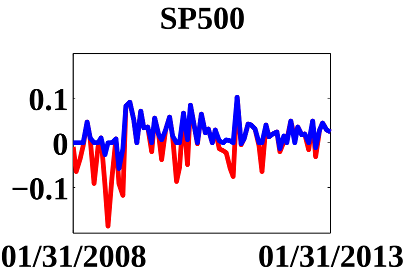

Figure 17: 73 return observations of a globally selected TEDAS portfolio (blue) versus the benchmark S&P500 index (red), Härdle et al. (2014)

▸ Financial Applications

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Small Sample

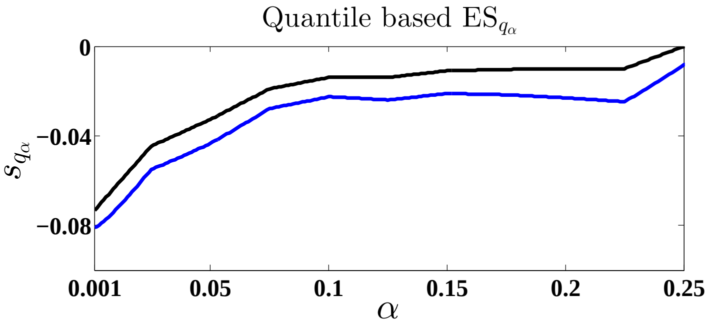

Figure 18: VaR_α (black) and quantile based ES_{q_α} using a normal scenario (blue) for the globally selected TEDAS portfolio

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Small Sample

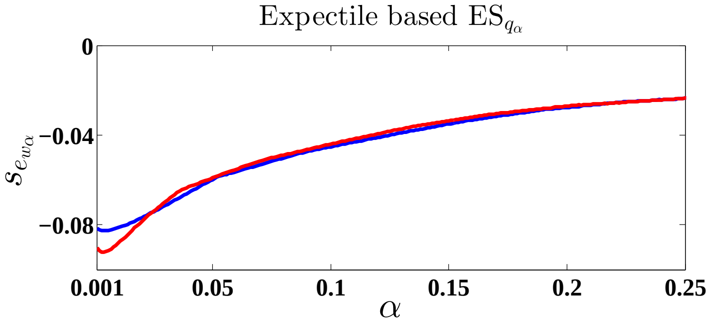

Figure 19: Expectile based ES_{q_α} using normal (blue) and $\frac{1}{3}$ Laplace contamination (red) scenario for the globally selected TEDAS portfolio

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Risk Corridor

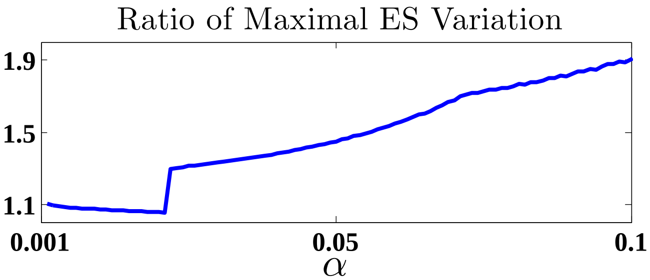

Figure 20: Ratio of the lowest and highest risk indication, i.e. maximal variation ratio, in a Normal-Laplace environment using the TEDAS sample

Financial Applications

TERES_RollingWindow

Conclusions

(i) Understanding Expected Shortfall (ES)

- Expectiles are successfully used for ES estimation
- Distributional families, mixtures

(ii) TERES

- ES_{q_α} for different risk levels α and scenarios
- Robustness of ES in a realistic financial setting

TERES - Tail Event Risk Expectile based Shortfall

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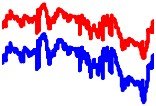

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Coherence

- Coherent risk measure $\rho(\cdot)$ of real-valued r.v.'s which model the returns
  - Subadditivity, $\rho(x+y) \geq \rho(x) + \rho(y)$ Details
  - Translation invariance, $\rho(x+c) = \rho(x)$ for a constant c
  - Monotonicity, $\rho(x) < \rho(y), \quad x < y$
  - Positive homogeneity, $\rho(kx) = k\rho(x), \quad k > 0$

VaR and ES

Subadditivity

Coherence

- $\rho(x+y) \leq \rho(x) + \rho(y)$
- Diversification never increases risk
- Quantiles are not subadditive
- Expected shortfall is subadditive, Delbaen (1998)

VaR and ES

ES using expectiles

- Expectile (see also 7-4 and 7-5)

$$e_\tau = \arg\min_\theta \mathsf{E}\,\rho_{\tau,2}\left(Y-\theta\right)$$

$$\rho_{\tau,2}\left(u\right) = \left|\tau - \mathbf{I}\left\{u<0\right\}\right|\left|u\right|^2$$

- First order condition

$$(1-\tau)\int_{-\infty}^{s}(y-s)f(y)dy - \tau\int_{s}^{\infty}(y-s)f(y)dy = 0$$

▸ Tail Structure

ES using expectiles

- Extension and reformulation

$$(1-\tau)\int_{-\infty}^{s}(y-s)f(y)dy-\tau\int_{-\infty}^{s}(y-s)f(y)dy$$
$$=\tau\int_{-\infty}^{\infty}(y-s)f(y)dy$$

- Rearranging

$$e_\tau - \mathsf{E}(Y) = \frac{1-2\tau}{\tau}\int_{-\infty}^{e_\tau}(y-e_\tau)f(y)dy$$

▸ Tail Structure

ES using expectiles

- Expected shortfall

$$e_\tau - \mathsf{E}[Y] = \frac{1-2\tau}{\tau}\,\mathsf{E}\left[(Y-e_\tau)\,\mathsf{I}\{Y<e_\tau\}\right]$$

$$\mathsf{E}[Y|Y<e_\tau] = e_\tau + \frac{\tau(e_\tau - \mathsf{E}[Y])}{(2\tau-1)F(e_\tau)}$$

- Use $e_{w(\alpha)} = q_\alpha$

$$\mathsf{E}[Y|Y<q_\alpha] = e_{w(\alpha)} + \frac{(e_{w(\alpha)} - \mathsf{E}[Y])w(\alpha)}{(2w(\alpha)-1)\alpha}$$

Tail Structure

Expectiles and Quantiles, $w(\alpha)$

- Relation of expectiles and quantiles (proof 7-7 and 7-8)

$$w(\alpha) = \frac{\text{LPM}_{e_{w(\alpha)}}(y) - e_{w(\alpha)}\alpha}{2\left\{\text{LPM}_{e_{w(\alpha)}}(y) - e_{w(\alpha)}\alpha\right\} + e_{w(\alpha)} - \text{E}[Y]}$$

- With the lower partial moment

$$\text{LPM}_u(y) = \int_{-\infty}^{u} yf(y)dy$$

Tail Structure | Expectiles and Quantiles

Expectiles and Quantiles, $w(\alpha)$

- Expectile (AND location estimate) solves

$$\{\alpha - 1\} \int_{-\infty}^{e_\alpha} (y - e_\alpha) f(y) dy = \underbrace{\alpha \int_{e_\alpha}^{\infty} (y - e_\alpha) f(y) dy}_{\pm \alpha \int_{-\infty}^{e_\alpha} (y - e_\alpha) f(y) dy}$$

- Rearrange

$$\alpha \left\{ e_\alpha - 2 \int_{-\infty}^{e_\alpha} e_\alpha f(y) dy \right\} + \int_{-\infty}^{e_\alpha} e_\alpha f(y) dy$$

$$= \alpha \left\{ \int_{-\infty}^{\infty} y f(y) dy - 2 \int_{-\infty}^{e_\alpha} y f(y) dy \right\} + \int_{-\infty}^{e_\alpha} y f(y) dy$$

▸ Tail Structure ▸ Expectiles and Quantiles

Expectiles and Quantiles, $w(\alpha)$

- Ordering terms

$$\alpha \left\{ 2 \left(\int_{-\infty}^{e_\alpha} y f(y) dy - e_\alpha \int_{-\infty}^{e_\alpha} f(y) dy \right) + e_\alpha - \mathsf{E}[Y] \right\}$$
$$= \int_{-\infty}^{e_\alpha} y f(y) dy - \int_{-\infty}^{e_\alpha} e_\alpha f(y) dy$$

- Solving for the risk level, $F(e_{w(\alpha)}) = \alpha$

$$w(\alpha) = \frac{\mathsf{LPM}_{e_{w(\alpha)}}(y) - e_{w(\alpha)}\alpha}{2\left\{\mathsf{LPM}_{e_{w(\alpha)}}(y) - e_{w(\alpha)}\alpha\right\} + e_{w(\alpha)} - \mathsf{E}[Y]}$$

▸ Tail Structure ▸ Expectiles and Quantiles

Expected Shortfall and Value at Risk

- Value at Risk (VaR)
  - For a cdf $F(\cdot)$ of an r.v. Y

$$\text{VaR}_\alpha = q_\alpha = F(\alpha)^{-1}$$

- Expected Shortfall (ES)
  - Basel: VaR threshold $\eta = q_\alpha$

$$ES_\eta = \text{E}[Y|Y < \eta]$$

VaR and ES

General Quantile Relation Level

- Solution z_α to (1)

$$\frac{\alpha}{1-\alpha} = \frac{\int_{-\infty}^{z_\alpha} |y - z_\alpha|^{\gamma-1}\, dF(y)}{\int_{z_\alpha}^{\infty} |y - z_\alpha|^{\gamma-1}\, dF(y)}$$

- Ordering terms

$$\alpha = \frac{\int_{-\infty}^{z_\alpha} |y - z_\alpha|^{\gamma-1}\, dF(y)}{\int_{\infty}^{\infty} |y - z_\alpha|^{\gamma-1}\, dF(y)}$$

Sensitivity Analysis